

Combinatorial Auctions with Budgets

Amos Fiat* Stefano Leonardi† Jared Saia ‡ Piotr Sankowski†§

April 21, 2010

“... *any color that he [the customer] wants so long as it is black.*”

— Henry Ford, *My Life and Work* (1922)

“... *Illogical approach to advertising budgets ...*”

— Michael Schudson, *Advertising, The Uneasy Persuasion: Its Dubious Impact on American Society* (1984)

Abstract

We consider budget constrained combinatorial auctions where bidder i has a private value v_i for each of the items in some set S_i , agent i also has a budget constraint b_i . The value to agent i of a set of items R is $|R \cap S_i| \cdot v_i$. Such auctions capture adword auctions, where advertisers offer a bid for those adwords that (hopefully) reach their target audience, and advertisers also have budgets. It is known that even if all items are identical and all budgets are public it is not possible to be truthful and efficient. Our main result is a novel auction that runs in polynomial time, is incentive compatible, and ensures Pareto-optimality. The auction is incentive compatible with respect to the private valuations, v_i , whereas the budgets, b_i , and the sets of interest, S_i , are assumed to be public knowledge. This extends the result of Dobzinski et al. [3, 4] for auctions of multiple *identical* items and public budgets to single-valued *combinatorial* auctions with public budgets.

*Tel Aviv University; Research partially supported by a grant from the Israel Academy of Sciences

†Sapienza University of Rome

‡University of New Mexico; Research was partially supported by NSF CAREER Award 0644058 and NSF CCR-0313160

§University of Warsaw

1 Introduction

In recent years ad auctions have been the subject of some non-negligible attention, perhaps because Internet ad revenue in 2009 was some $\$2.4 \times 10^{10}$ USD¹. Much practical and theoretical work has been done on the issue of ad auctions, much of this work within the general framework of mechanism and auction design. If all advertisers bid for (multiple) copies of a single search term, (so called “multi unit auction”), then — the Vickrey multi unit auction [7] is both truthful and maximizes efficiency.

The Vickrey multi unit auction is not entirely satisfactory, partially because of the following:

1. Budgets - budgets are a necessary evil because of limited resources and risk aversion. In any real system, budgets are a key component. The Vickrey multi unit auction is not incentive compatible when budgets are allowed. Moreover, even efficiency is ill defined in such a setting (the next best thing is Pareto-optimality).
2. Not all items are equal, typically, if you want to sell precious metals you probably want to advertise on search terms “Gold”, “Silver”, “Platinum” and not “Lead” or “Corn”. If you sell all metals excluding Silver and Platinum then you may want to advertise on search terms “Gold”, “Uranium”, “Plutonium”, and “Lead”. Multiple parallel multi unit auctions, one for each and every search term, are somewhat problematic and certainly not strategy proof.

One real system that addresses both issues is Google’s Auction for TV Ads, deployed a few years ago [6]. This auction allows bidders to select shows, times, and days they wish to advertise on; and then give a per-ad impression bid and a total budget. The theoretical analysis of Google’s TV ad auction is yet incomplete, but it is known not to be incentive compatible — strategic bidders can gain by misrepresentation of their valuation (the Google system does not allow one to choose different valuations for different ad slots), even if all other bidder parameters are public.

Much of the theoretical work on mechanism design has ignored budgets. This may be because budgets mean that utilities are not quasi-linear, the Vickrey-Clarke-Groves (VCG) mechanism is not incentive-compatible, and other curiosities.

A seminal paper on mechanisms for ad auctions with budgets is by Dobzinski, Lavi and Nisan [3, 4]. They consider multi unit auctions (all items are identical). E.g., multiple occurrences of the same search word. Dobzinski et. al. give an incentive-compatible auction (with respect to valuation) that produces a Pareto-optimal allocation. This result holds if one assumes that the budgets are public information and [3, 4] also show that this assumption is required: there is no incentive-compatible auction with respect to both valuation and budgets that produces a Pareto-optimal allocation.

Subsequently, Aggarwal, Muthukrishnan, Pal and Pal [1] considered the case where bidders seek at most one item — not quite relevant for ad auctions. In this setting they give an incentive compatible auction, with respect to both valuation and budgets. This latter result is related to the paper of Hatfield and Milgrom [5] who consider more general non-quasi-linear utilities. Both [1] and [5] are in a more general combinatorial setting where agents are interested in a given subset of items, or may even can have different valuations for items.

Our work here seeks to map out the frontier of the possible. We give incentive compatible combinatorial auctions with budgets that produce Pareto-optimal allocations, for some not entirely general but also non-trivial class of auctions (the same class considered in Google’s TV ad auction). Furthermore, we show that these restrictions cannot be circumvented. Thus, arguably, what we do here is the most that can be done, given that we require that the allocation is Pareto-optimal.

¹According to the Interactive Advertising Bureau, www.iab.net.

In this paper we study combinatorial auctions of the following **general form**:

- Every agent (bidder) $1 \leq a \leq n$ has a *publicly known* budget, $b_a \geq 0$, and an unknown (private) valuation $v_a > 0$;
- Every agent a is “interested” in some *publicly known* set of items, S_a . We assume that there is at least one agent interested in every item. Agent a is allocated some (possibly empty) subset of S_a .
- The auction produces an allocation (M, P) . $M \subseteq \{1, \dots, n\} \times \{1, \dots, m\}$ is a (partial) matching between agents (bidders) and items. $P \in \mathbb{R}^n$ is a vectors of payments made by the agents. For agent $1 \leq a \leq n$, let M_a be the number of items sold to agent a over the course of the auction and P_a be the total payment made by agent a during the course of the auction. The allocation must obey the following conditions:
 1. The payment by agent a , P_a , cannot exceed the budget b_a .
 2. The utility for agent $1 \leq a \leq n$ is $u_a = M_a v_a - P_a$.
 3. The utility for the auctioneer is $\sum_{j=1}^n P_j$.
 4. Bidder-rationality: for all agents $1 \leq a \leq n$, $u_a \geq 0$.
 5. Auctioneer-rationality: the utility of the auctioneer, $\sum_{j=1}^n P_j \geq 0$.

Note² that auctioneer-rationality is implied by *no positive transfers*: $P_a \geq 0$ for all $1 \leq a \leq n$.

Given valuations, v_a , budgets, b_a , and sets of interest, S_a , we define (M, P) to be *Pareto-optimal* if there is no other allocation (M', P') such³ that

1. The utility of every bidder in (M, P) is not less than the utility in (M', P') , and
2. The utility of the auctioneer in (M, P) is not less than the utility in (M', P') , and
3. At least one bidder or the auctioneer is better off in (M', P') compared with (M, P) .

An auction is said to be incentive compatible if it is a dominant strategy for all bidders to reveal their true valuation. An auction is said to be Pareto-optimal if the allocation it produces is Pareto-optimal. An auction is said to make no positive transfers if the allocation it produces has no positive transfers.

When the sets S_a consist of all items for all agents, i.e., all items are identical, Dobzinski, Lavi, and Nisan [3, 4] show that there are no incentive compatible mechanisms that are Pareto-optimal when both valuations and budgets are private. Furthermore, they also show that a version of Ausubel’s dynamic clinching multi-unit auction [2] is truthful and Pareto-optimal for agents with budgets, *when budgets are public knowledge*.

2 Our Results

In this paper we give an incentive compatible and Pareto-optimal combinatorial auction.

Furthermore, our auction makes no positive transfers.

Our result can be viewed as extending the results of [3] from selling off multiple *identical* items to a new combinatorial setting where items are distinct and different agents may be interested in different items. In particular, for the non-combinatorial multi unit setting of [3, 4], our auction and the auction of [3, 4] produce the same allocation. That said, we claim that our version, when restricted to the simpler multi unit setting, is much easier to follow⁴.

²In [4] the authors refer to what we call auctioneer rationality by the term “weakly no positive transfers”.

³Note that no restrictions are placed on the matching M' or on the payments P' .

⁴Karl Popper would say that this claim cannot be falsified.

Our combinatorial auction is polynomial time and deterministic. Obviously, this cannot be if we were to consider the full generality of combinatorial auctions. We consider combinatorial auctions where agents have an agent-specific set of interesting items, but only one valuation for any item from that set of interest.

In light of the impossibility results of Dobzinski et al. [3] we could not hope to achieve this result with private budgets. We further show that public budgets alone are insufficient for Pareto-optimality and incentive compatibility. We prove that one cannot avoid the restrictions we place on the combinatorial auction setting in the following sense:

- if budgets are public but the sets of interest and the valuations are private then no truthful Pareto-optimal auction is possible;
- if budgets are public and private arbitrary valuations are allowed, no truthful and Pareto-optimal auction is possible (irrespective of computation time). This follows by simple reduction to the previous claim on private sets of interest.

In Section 3 we present our mechanism. It is straightforward to show that the mechanism is truthful with respect to valuations. However, it is not trivial to prove that the mechanism is Pareto optimal. In Section 4 we prove that the allocation produced by the mechanism is in fact Pareto optimal. In Section 5 we complement our positive result by showing that with public budgets, private valuations, and private sets of interest, there can be no truthful Pareto optimal mechanism.

3 Combinatorial Auctions with Budgets via Dynamic Clinching

In this section we describe our mechanism in detail.

Our auction can be implemented as a direct revelation mechanism (where the agents reveal their private types to the mechanism) but may also be viewed as an incentive compatible ascending auction (where incentive compatible means ex-post Nash). The ascending auction raises the price of unsold items till all items are clinched. We describe the mechanism as a direct revelation mechanism and assume that the private value \tilde{v}_a is equal to the bid v_a . The details of the mechanism are presented in Algorithm 1, Algorithm 2 and Algorithm 3.

Throughout the algorithm there is always some current price p (initially zero), current number of unsold items, m (initially equal to total number of items), and current remaining budgets $b = (b_1, b_2, \dots, b_n)$, where b_a is the remaining budget for agent $1 \leq a \leq n$. In addition, the algorithm maintains a boolean vector $H = (H_1, H_2, \dots, H_n)$.

For every agent $1 \leq i \leq n$ the mechanism makes use of values D_i , D_i^+ , and d_i , these values are functions of the current values of p , m , b , and H . I.e., whenever one of these values is referenced it is computed based upon the current values of p , m , b , and H . Later on, we omit these arguments in the description of the mechanism. Formally:

$$D_i = D_i[p, b_i, m] = \begin{cases} \min\{m, \lfloor b_i/p \rfloor\} & \text{if } p \leq v_i \\ 0 & \text{if } p > v_i \end{cases} \quad (1)$$

$$D_i^+ = D_i^+[p, b_i, m] = \lim_{\epsilon \rightarrow 0^+} D_i[p + \epsilon, b_i, m]; \quad (2)$$

$$d_i = d_i[p, b_i, m, H] = \begin{cases} D_i & \text{if } H_i = \text{True} \\ D_i^+ & \text{if } H_i = \text{False} \end{cases} \quad (3)$$

D_i is equal to the number of items that agent i is interested in purchasing at current p , m , and b (Equation 1). In Equation 2 we define D_i^+ , what is equal to the number of items that agent a_i would be interested in purchasing if the price were increased by an infinitesimally small amount, thus $D_i^+ \leq D_i$. In Equation (3) we define d_i , the current demand of agent i , d_i is either equal to D_i or to D_i^+ , depending on the value of H_i .

The algorithm also implicitly keeps a set of unsold items U (those items not yet sold in Algorithm 3), a set of active agents A — those with current demand greater than zero, and a set of value limited agents V — those with valuation equal to the current price:

$$A = \{1 \leq a \leq n \mid d_a > 0\}, \quad (4)$$

$$V = \{1 \leq a \leq n \mid d_a > 0, v_a = p\}. \quad (5)$$

A key tool used in our auction is that of S -avoid matchings. These are maximal matchings that try to avoid, if at all possible, assigning any items to bidders in some set S . Such a matching can be computed by computing a min cost max flow, where there is high cost to direct flow through a vertex of S .

In general, the auction prefers to sell items only at the last possible moment (alternately phrased, the highest possible price) at which this item can still be sold while still preserving incentive compatibility. The auction will in fact sell all items (Lemma 3.1).

Once a price has been updated, the auction checks to see if it must sell items to value limited bidders. Such bidders will receive no real benefit from the item (their valuation is equal to their payment), but this is important so as to increase the utility of the auctioneer. Our definition of Pareto-optimality includes all bidders and the auctioneer. To check if this is indeed the case, the auction computes a V -avoid matching, trying to avoid the bidders in V . If this cannot be done, then items are sold to these V bidders. After items are sold to value limited bidders, these bidders effectively disappear by setting their H_a values to False.

The main loop of the mechanism checks whether any items must be sold to any of the currently active bidders. This is where incentive compatibility comes into play. The auction sells an item to some bidder, a , at the lowest price where the remaining bidders total demand is such that an item can be assigned to a without creating a shortage. Again, this makes use of the $\{a\}$ -avoid matching, if in the $\{a\}$ -avoid matching some item is matched to a then a must be sold that item.

If no items can be sold in this manner, the demand of the bidders is reduced by setting H_a to False, for some active bidder a . When neither action can be done, the price increases.

The following lemma shows that all items will in fact be sold.

Lemma 3.1 *If every item appears in $\cup_{i=1}^n S_i$ then the auction will sell all items.*

Proof in Appendix A.

4 Pareto-Optimality of the Combinatorial Auction with Budgets

Definition 4.1 *An allocation (M, P) is Pareto-optimal if for no other allocation (M', P') are all players better off, $M'_i v_i - P'_i \geq M_i v_i - P_i$, including the auctioneer $\sum_i P'(i) \geq \sum_i P_i$, with at least one of the inequalities strict.*

The main goal of this section is to prove the following theorem:

Algorithm 1 Combinatorial Auction with Budgets

```
1: procedure COMBINATORIAL AUCTION WITH BUDGETS( $v, b, \{S_i\}$ )
   Implicitly defined  $D_a, D_a^+, d_a, U, A$ , and  $V$  — see Equations (1) – (5).
    $B(\neg\{a\})$  - number of items assigned to agents in  $A \setminus \{a\}$  in  $\{a\}$ -avoid matching
2:    $p \leftarrow 0$ 
3:   while ( $A \neq \emptyset$ ) do
4:      $\forall a \in A : H_a \leftarrow \text{True}$ 
5:      $\text{Sell}(V)$ 
6:      $\forall a \in V : H_a \leftarrow \text{False}$ 
7:     repeat
8:       if  $\exists a | B(\neg\{a\}) < m$  then  $\text{Sell}(a)$ 
9:       else
10:        For arbitrarily  $a \in A$  with  $H_a = \text{True}$  set  $H_a \leftarrow \text{False}$ 
11:      end if
12:    until  $\forall a \in A : (\neg H_a)$  and  $(B(\neg\{a\}) \geq m)$ 
13:    Increase  $p$  until for some  $a \in A$ ,  $D_a^+$  changes (decreases)
14:  end while
15: end procedure
```

Algorithm 2 Computing an avoid matching, can be done via min cost max flow

```
1: procedure  $S$ -AVOID MATCHING
   Construct interest graph  $G$ :
   • Active agents,  $A$ , on left, capacity constraint of agent  $a \in A = d_a$ 
   • Unsold items,  $U$ , on right, capacity constraint 1.
   • Edge  $(a, t)$  from agent  $a \in A$  to unsold item  $t \in U$  iff  $t \in S_a$ .
   Return maximal  $B$ -matching with minimal number of items assigned to agents in  $S$ ,
   amongst all maximal  $B$ -matchings.
2: end procedure
```

Algorithm 3 Selling to a set S

```
1: procedure  $\text{SELL}(S)$ 
2:   repeat
3:     Compute  $Y = S$ -AVOID MATCHING
4:     For arbitrary  $(a, t)$  in  $Y$ ,  $a \in S$ , sell item  $t$  to agent  $a$ .
5:   until  $B(\neg S) \geq m$ 
6: end procedure
```

Theorem 4.2 *The allocation (M^*, P^*) produced by Algorithm 1 is Pareto-optimal. Moreover, the mechanism makes no positive transfers.*

In Section 4.1 we define the notion of trading paths and show the equivalence between allocations with no trading paths and Pareto optimal allocations. In Appendix C we attempt to give some intuition as to why these two are related as well as to why Theorem 4.2 gives the desirable outcome. In Section 4.2 we show that the final allocation produced by Algorithm 1 contains no trading paths, thus concluding the proof of Theorem 4.2.

4.1 Alternating paths, Trading paths, and Pareto-optimality

Definition 4.3 *Consider a path $\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j)$, in a bipartite graph G . We say that the path π is an alternating path with respect to B -matching M if $(a_i, t_i) \in M$ and $t_i \in S_{i+1}$ for all $1 \leq i < j$. We say that an alternating path is simple if no agent appears more than once along the path. Note that all alternating paths are of even length (even number of edges).*

Definition 4.4 *A path $\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j)$ is called a trading path with respect to the allocation (M, P) if the following hold:*

1. π is a simple alternating path with respect to M , (which implies that agent a_i , $i < j$, was allocated item t_i during the course of the auction).
2. The valuation of agent a_j , v_{a_j} is strictly greater than the valuation of agent a_1 , v_{a_1} .
3. The remaining (unused) budget of agent a_j at the conclusion of the auction, $b_{a_j}^*$, is \geq the valuation of agent a_1 , v_{a_1} .

Intuitively, trading paths, as their name suggests, represent possible trades amongst agents. A trading path allows a trade to take place, where the endpoints of the trading path are better off following the trade, and the interior agents no worse off. (In fact, they can all be made better off by paying a “commission” of sorts along the path).

We now turn to the following equivalence:

Theorem 4.5 *Any allocation (M, P) is Pareto-optimal⁵ if and only if*

1. All items are sold in (M, P) , and
2. There are no trading paths in G with respect to (M, P) .

Proof in Appendix B.

4.2 No Trading Paths in (M^*, P^*)

To conclude the proof of Theorem 4.2 we now prove that there are no trading paths in the final allocation (M^*, P^*) generated by the mechanism given in Algorithm 1.

We know from Lemma 3.1 that M^* matches all items.

Consider the set of all trading paths Π in the final allocation M^* .

Definition 4.6 *We define the following for every $\pi \in \Pi$:*

⁵We remark that an analogous (but simpler) claim made in the proceedings version of the multi unit auction with budget paper [3] was incorrect but was corrected in [4].

- Let Y^π be the S -avoid matching used the first time some item t is sold to some agent a where (a, t) is an edge along π . Y^π is either a V -avoid matching (line 5 of Algorithm 1) or an a -avoid matching for some agent-item edge (a, t) along π (line 8 of Algorithm 1).
- If Y^π is a V -avoid matching, let V^π be this set of value limited agents.
- If Y^π is an a -avoid matching, let a^π be this agent.
- Let $F^\pi \subset M^*$ be the set of edges (a, t) such that item t was sold to agent a at or subsequent to the first time that some item t' was sold to some agent a' for some edge $(a', t') \in \pi$ ((a', t') is itself in F^π).
- Let m^π be the number of unsold items just before the first time some edge along π was sold. I.e., m^π is equal to the number of items matched in F^π .
- Let p^π be the price at which item[s] were sold from Y^π .
- Let b_a^π be the remaining budget for agent a before any items are sold in $\text{Sell}(V^\pi)$ or $\text{Sell}(a^\pi)$.

We partition Π into two classes of trading paths:

1. Π_V is the set of trading paths such that $\pi \in \Pi_V$ iff Y^π is some V^π -avoid matching used in $\text{Sell}(V^\pi)$ (line 5 of Algorithm 1).
2. $\Pi_{\neg V}$ is the set set of trading paths such that $\pi \in \Pi_{\neg V}$ iff Y^π is some a^π -avoid matching used in $\text{Sell}(a^\pi)$ (line 8 of Algorithm 1).

Lemma 4.7 $\Pi_V = \emptyset$.

Proof:

We need the following Claim:

Claim 4.8 *Given a trading path $\pi = (a_1, t_2, \dots, a_{j-1}, t_{j-1}, a_j) \in \Pi_V$, and let (a_i, t_i) be the last edge belonging to Y^π along π . Then the suffix of π starting at a_i , (a_i, t_i, \dots, a_j) , is itself a trading path.*

Proof: This trivially follows as the valuation of a_i is equal to current price when $\text{Sell}(V^\pi)$ was done (p^π), and the valuation of a_1 is $\geq p^\pi$ as edge (a_1, t_1) was unsold prior to this $\text{Sell}(V^\pi)$ and does belong to the final F^π . \square

From the Claim above we may assume, without loss of generality, that if $\Pi_V \neq \emptyset$ then $\exists \pi \in \Pi_V$ such that the first edge along π was also the first edge sold amongst all edges of π , furthermore, all subsequent edges do not belong to Y^π .

As agents $a \in V^\pi$ will not be sold any further items after this $\text{Sell}(V^\pi)$, the items assigned to a_1 in Y^π are the same items assigned to a_1 in F^π .

We seek a contradiction to the assumption that Y^π was a V^π -avoid matching. Note that the matching F^π is a V^π -avoid matching by itself, because exactly the items assigned to V -type agents in Y^π are sold. We now show how to construct from F^π another matching that assigns less items to V -type agents.

We show that the number of items assigned to agent a_1 in F^π (which is the same as in Y^π) can be reduced by one by giving agent a_{k+1} item t_k for $k = 1, \dots, j-1$. This is also a full matching but it remains to show that this does not exceed the capacity constraints for agent a_j , d_{a_j} .

As $H_a = \text{True}$ for all $a \in A$ when $\text{Sell}(V^\pi)$ is done, this means that $d_{a_j} = D_{a_j}$. Agent a_j has remaining budget $\geq v_1$ at the conclusion of the auction, and all items assigned to agent a_j in F^π are at price $\geq p^\pi = v_1$. This implies that at the time of $\text{Sell}(V^\pi)$ we have $D_{a_j} >$ the number of

items assigned to a_j in F^π . Thus, we can increase the number of items allocated to a_j by one without exceeding the demand constraint $d_{a_j} = D_{a_j}$.

Now, note that a_j is not V -type agent, so the new matching constructed assigns less items to V type agents than the matching F^π . Hence, F^π is not an V^π -avoid matching, and in turn neither Y^π is V^π -avoid matching. \square

We've shown that $\Pi_V = \emptyset$. It remains to show that $\Pi_{-V} = \emptyset$.

Assume $\Pi_{-V} \neq \emptyset$. Order $\pi \in \Pi_{-V}$ by the first time at which some edge along π was sold. We know that this occurs within some $\text{Sell}(a^\pi)$ for some a^π and that $a^\pi \notin V$. Let us define $\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j)$ be the last path in this order, and let $e = (a^\pi, t^\pi) = (a_i, t_i)$.

Recall that Y^π is the a^π -avoid matching used when item t^π was sold to agent a^π . Also, $F^\pi \subset M^*$ is the set of edges added to M^* in the course of the auction from this point on (including the current $\text{Sell}(a_i)$).

Lemma 4.9 *Let π , $a^\pi = a_i$, $t^\pi = t_i$, be as above, we argue that when Y^π was computed as an a^π -avoid matching there was another full matching X with the following properties:*

1. *The suffix of π from a_i to a_j :*

$$\pi[a_i, \dots, a_j] = (a_i, t_i, a_{i+1}, t_{i+1}, \dots, a_{j-1}, t_{j-1}, a_j),$$

is an alternating path with respect to X . (I.e., edges (a_k, t_k) , $i \leq k \leq j-1$, belong to X).

2. *The number of items assigned to a_i in X is equal to the number of items assigned to a_i in Y^π .*
3. *The number of items assigned to a_j in X is equal to the number of items assigned to a_j in F^π .*

Proof: Consider the final matching F^π . Note that $F^\pi(a_i) \geq Y^\pi(a_i)$, because otherwise if $F^\pi(a_i) < Y^\pi(a_i)$ then $F^\pi(a_i)$ would have fewer items assigned to a_i than the a_i -avoid matching Y^π , a contradiction.

If $F^\pi(a_i) = Y^\pi(a_i)$ then choose $X = F^\pi$ and conditions 1 – 3 above hold trivially.

Thus, we are left with the case where $F^\pi(a_i) > Y^\pi(a_i)$. Consider the symmetric difference $F^\pi \oplus Y^\pi$. By Lemma B.1 the edges of $F^\pi \oplus Y^\pi$ can be covered by alternating paths with respect to F^π . There must be $\delta = F^\pi(a_i) - Y^\pi(a_i)$ such paths starting at agent a_i (as agent a_i has δ more items assigned in F^π than in Y^π). Take one of these paths $\tau = (a_i = g_1, s_1, g_2, s_2, \dots, g_\ell)$, g_k 's are agents, s_k 's are items, (g_k, s_k) belongs to F^π , (s_k, g_{k+1}) belongs to Y^π .

We now argue that τ and $\pi[a_i, \dots, a_j]$ are vertex disjoint besides the first agent a_i . To reach a contradiction, assume that there is another common vertex u along τ and along $\pi[a_i, \dots, a_j]$, $u \neq a_i$. Choose u to be the first such vertex along τ .

We consider two possibilities:

1. u is an item. Consider

$$\pi[a_i, \dots, a_j] = (a_i, t_i, a_{i+1}, t_{i+1}, \dots, a_{j-1}, t_{j-1}, a_j),$$

and let $u = s_k = t_{k'}$ for some k, k' . Then both $(g_k, s_k = t_{k'} = u)$ and $(a_{k'}, s_k = t_{k'} = u)$ belong to F^π . This implies either that item u is assigned to two different agents in F^π or that $a_{k'} = g_k$ in contradiction to our choice of u as the first common vertex along τ .

2. u is an agent. For some $i < k \leq j$, $1 < k' \leq \ell$, $u = g_k = a_{k'}$. Let π' be the concatenation of the prefix of π up to a_i , followed by the prefix of τ up to g_k and then followed by the suffix of π from $g_k = a_{k'}$ to the end:

$$\pi' = (a_1, t_1, \dots, a_i = g_1, s_1, g_2, \dots, g_k = a_{k'}, t_{k'}, a_{k'+1}, \dots, a_j).$$

This path is a trading path in F^π , and none of the edges along this path were sold before the edge (a_i, t_i) , in contradiction to the assumption that π had it's first sold edge sold last amongst all trading paths.

Therefore, τ and $\pi[a_i, \dots, a_j]$ only have a_i in common. By Lemma B.1 the different paths τ starting from a_i in $Y^\pi \oplus F^\pi$ are edge disjoint. For any such $\tau = (a_i = g_1, s_1, g_2, s_2, \dots, g_\ell)$, agent g_k holds item s_k in F^π , $1 \leq k \leq \ell - 1$, and agent g_{k+1} holds item s_k in Y^π , $1 \leq k \leq \ell - 1$. Therefore, we can move item s_k from agent g_k to agent g_{k+1} , $1 \leq k \leq j - 1$, without violating the demand of agent g_ℓ because $s_{\ell-1}$ was assigned to g_ℓ in Y^π . As we can do so for all such paths τ we obtain a new full matching X where the number of items assigned to agent a_i is the same as the number of items assigned to agent a_i in Y^π .

Note that, other than a_i , none of the agents along the path $\pi[a_i, \dots, a_j]$ appears on any of these τ and therefore their assignment in X remains unchanged from their assignment in F^π . \square

Corollary 4.10 $\Pi_{-V} = \emptyset$.

Proof: Assume $\pi \in \Pi_{-V} \neq \emptyset$ and let $a^\pi = a_i$, $t^\pi = t_i$, we now seek to derive a contradiction as follows:

- When Y^π was computed there was also an alternate full matching Y' with fewer items assigned to agent a_i , contradicting the assumption that Y^π is an a_i avoid matching. Or,
- We show that the remaining budget of agent a_j at the end of the auction, $b_{a_j}^*$, has $b_{a_j}^* < v_1$, contradicting the assumption that π is a trading path.

Let X be a matching as in Lemma 4.9 and F^π be as defined in Definition 4.6. Also, let $X(a)$, $F^\pi(a)$, be the number of items assigned to agent a in full matchings X , F^π , respectively.

We consider the following cases regarding d_{a_j} when Y^π , the a_i -avoid matching, was computed:

1. $d_{a_j} > X(a_j)$: then, like in Lemma 4.7, we can decrease the number of items sold to a_i by assigning item t_k to agent a_{k+1} for $k = i, \dots, j - 1$, without exceeding the d_{a_j} demand constraint.
2. $d_{a_j} = X(a_j)$, by subcase analysis we show that $b_{a_j}^\pi \leq (X(a_j) + 1)p^\pi$:
 - (a) $D_{a_j} = D_{a_j}^+$: Observe that $X(a_j) < m$, the current number of unsold items. This follows because $X(a_i) = Y^\pi(a_i) \geq 1$ by assumption that t_i was assigned to a_i in Y^π . This means that $d_{a_j} = X(a_j) < m$ so

$$\begin{aligned} X(a_j) &= d_{a_j} = \left\lfloor b_{a_j}^\pi / p^\pi \right\rfloor > b_{a_j}^\pi / p^\pi - 1 \\ \Rightarrow b_{a_j}^\pi &< (X(a_j) + 1)p^\pi. \end{aligned}$$

- (b) $D_{a_j} \neq D_{a_j}^+$: Observe that $a_j \notin V$ as $v_{a_j} > v_{a_i}$ and $a_i \notin V$. As $a_j \notin V$, the only reason that $D_{a_j} \neq D_{a_j}^+$ is because the remaining budget of agent a_j , $b_{a_j}^\pi$, is an integer multiple of the

current price p^π . Then, $D_{a_j}^+ = D_{a_j} - 1$ and $D_{a_j} = \lfloor b_{a_j}^\pi / p^\pi \rfloor = b_{a_j}^\pi / p^\pi$, it follows that

$$\begin{aligned} X(a_j) &= d_{a_j} \geq D_{a_j}^+ = D_{a_j} - 1 = b_{a_j}^\pi / p^\pi - 1 \\ \Rightarrow b_{a_j}^\pi &\leq (X(a_j) + 1)p^\pi. \end{aligned}$$

Note that the current price $p^\pi < v_{a_i}$ because we assume that a_i was sold t_i as a result of $\text{Sell}(a_i)$ and not $\text{Sell}(V)$. It is also true that $v_{a_i} \leq v_{a_1}$ as (a_i, t_i) was the first edge that was sold along π . By condition 3 of Lemma 4.9 we can deduce that

$$b_{a_j}^\pi \leq (X(a_j) + 1)p^\pi = (F^\pi(a_j) + 1)p^\pi.$$

Agent a_j is sold exactly $F^\pi(a_j)$ items at a price not lower than p^π , to at the end of the auction the remaining budget for agent a_j , $b_{a_j}^*$, is $\leq p^\pi$. This contradicts the assumption that π is a trading path since

$$b_{a_j}^* \leq p^\pi < v_{a_i} \leq v_{a_1}.$$

□

5 Mapping the Frontier

In this paper we gave a mechanism that is incentive compatible with respect to valuation, and produces a Pareto-optimal allocation, but with various annoying restrictions and assumptions:

- we assume public budgets;
- we assume public sets of interest;
- moreover, agents are restricted to have a step function valuation for items, if the item is in S_i then its valuation is v_i , otherwise zero.

This poses the question: can we remove these annoying assumptions/restrictions? Just how far can we go?

As for private budgets, it was shown by [3] that even for the multi unit case, one cannot achieve incentive compatibility with respect to valuation along with bidder rationality, auctioneer rationality, and obtain a Pareto-optimal allocation.

We argue that even if one assumes public budgets, the other restrictions are also necessary. This is summarized in the following theorems:

Theorem 5.1 *There is no truthful, bidder rational, auctioneer rational and Pareto-optimal auction with public budgets, b_a , private valuations, v_a , and private sets of interest, S_a .*

Proof in Appendix D.

Corollary 5.2 *There is no truthful, bidder rational, auctioneer rational and Pareto-optimal auction with public budgets, b_a , and private item-dependent valuations v_{at} .*

Proof: This follows immediately from Theorem 5.1. Consider the case where the private valuations v_{at} are zero for any $t \notin S_a$, and v_a for $t \in S_a$. □

References

- [1] Gagan Aggarwal, S. Muthukrishnan, Dávid Pál, and Martin Pál. General auction mechanism for search advertising. In *WWW '09: Proceedings of the 18th international conference on World wide web*, pages 241–250, New York, NY, USA, 2009. ACM.
- [2] Lawrence M. Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94(5):1452–1475, December 2004.
- [3] Shahrar Dobzinski, Ron Lavi, and Noam Nisan. Multi-unit auctions with budget limits. In *FOCS*, pages 260–269. IEEE Computer Society, 2008.
- [4] Shahrar Dobzinski, Ron Lavi, and Noam Nisan. Multi-unit auctions with budget limits, 2010.
- [5] J. Hatfield and P. Milgrom. Matching with contracts. *The American Economic Review*, 95(4):913–935, 2005.
- [6] Noam Nisan, Jason Bayer, Deepak Chandra, Tal Franji, Robert Gardner, Yossi Matias, Neil Rhodes, Misha Seltzer, Danny Tom, Hal R. Varian, and Dan Zigmund. Google’s auction for tv ads. In Susanne Albers, Alberto Marchetti-Spaccamela, Yossi Matias, Sotiris E. Nikolettseas, and Wolfgang Thomas, editors, *ICALP (2)*, volume 5556 of *Lecture Notes in Computer Science*, pages 309–327. Springer, 2009.
- [7] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, 16(1):8–37, 1961.

Notation	Explanation
n	Number of agents
m	Current number of items
S_a	Items agent a is interested in
$v \in \mathbb{R}^m$	$v_a > 0$ is the valuation of agent a for the items in S_a
$b \in \mathbb{R}^m$	b_a is the current budget for agent a
$p \in \mathbb{R}^+$	The current price
A	Current active agents ($d_a > 0$)
V	Current value limited agent ($d_a > 0, v_a = p$)
U	Current set of unsold items
D_a	$\begin{cases} \min\{m, \lfloor b_i/p \rfloor \} & \text{if } p \leq v_i \\ 0 & \text{if } p > v_i \end{cases}$
D_a^+	D_a at infinitesimally higher price than p
d_a	D_a if $H_a = \text{True}$, D_a^+ otherwise
H_a	Boolean value, if true $d_a = D_a$, OW $d_a = D_a^+$
(M^*, P^*)	The matching and payments resulting from the auction
M_i	The number of items sold to agent i in matching M
P_i	The total payment by agent i given payment vector $P \in \mathbb{R}^n$
Π	The set of all trading paths in M^*
$\pi \in \Pi$	A trading path $(a_1, t_1, \dots, a_{j-1}, t_{j-1}, a_j)$
$\pi[a_i, \dots, a_j]$	A suffix of π : (a_i, t_i, \dots, a_j)
V^π	First time any edge was sold from π was during $\text{Sell}(V^\pi)$
a^π	First time any edge was sold from π was during $\text{Sell}(a^\pi)$
Y^π	Either V^π -avoid matching or a^π -avoid matching
Π_V	First time any edge was sold from $\pi \in \Pi_V$ was during $\text{Sell}(V^\pi)$
$\Pi_{\neg V}$	First time any edge was sold from $\pi \in \Pi_{\neg V}$ was during $\text{Sell}(a^\pi)$
b_a^π	Budget of agent a before 1st time any edge sold from π
b_a^*	Remaining budget of agent a at end of auction
$B(\neg S)$	# items assigned to agents in $A \setminus S$ in S -avoid matching

Table 1: Notation Used

A Proof of Lemma 3.1

Proof: We prove that throughout the auction, there is always a matching that can sell all remaining items at the current price without exceeding the budget of any agent. As prices only increase, eventually all items must be sold. The lines below refer to Algorithm 1 unless stated otherwise.

Initially, all items can be sold at price zero. The d_a capacity constraints are all equal to m .

Furthermore, we argue that it is always true that all unsold items can be sold to active agents at the current price without violating the capacity constraints. We prove this invariant by case analysis of the following events:

- Increase in price followed by setting the H_a variables to True: The repeat loop in lines 7 – 12 ends with $H_a \leftarrow \text{False}$ and $B(\neg\{a\}) \geq m$ for all agents a . Thus, when the condition in line 12 is met, all the d_a 's are set to D_a^+ .

Any increment in price in line 13 will set D_a equal to the previous D_a^+ and the subsequent assignment of $H_a \leftarrow \text{true}$ (line 4) means that the new d_a 's are equal to the old ones. Thus, any matching valid at the old price is valid at the new price.

- The $\text{Sell}(V)$ operation (line 5 of Algorithm 1, Algorithm 3) sells items to agents in V only if all other unsold items can be matched to agents not in V .
- Setting $H_a \leftarrow \text{False}$ for $a \in V$ (line 6) sets $d_a = 0$ for $a \in V$ and this is OK because nothing will be sold to $a \in V$ at any higher price.
- The $\text{Sell}(a)$ operation (line 8 of Algorithm 1, Algorithm 3) sells items to agent a only if all other unsold items can be matched to other agents.
- Setting $H_a \leftarrow \text{False}$ (line 10) is done only if $B(\neg\{a\}) \geq m$, i.e., all unsold items can be matched to the other agents (not including a).

Thus, the mechanism will sell all items. \square

B Proof of Theorem 4.5

Proof: Let Q be the predicate that (M, P) is Pareto-optimal, R_1 be the predicate that all items are sold in (M, P) , and R_2 the predicate that there are no trading paths in G with respect to (M, P) . We seek to show that $Q \Leftrightarrow R_1 \cap R_2$.

$Q \Rightarrow (R_1 \cap R_2)$: to prove this we show that $(\neg R_1 \cup \neg R_2) \Rightarrow \neg Q$.

If both R_1 and R_2 are true then this becomes $\text{False} \Rightarrow Q$ which is trivially true.

If the allocation (M, P) does not assign all items ($\neg R_1$) then it is clearly not Pareto-optimal ($\neg Q$). We can get a better allocation by assigning all unsold items to any agent i with such items in S_i . This increases the utility of agent i .

If $\neg R_2$ then there exists a trading path in G with respect to (M, P) , let this path be $\pi = (a_1, t_1, a_2, t_2, \dots, a_{j-1}, t_{j-1}, a_j)$, as $v_{a_j} > v_{a_1}$ and $b_{a_j}^* \geq v_{a_1}$ then we can decrease the payment of agent a_1 by v_{a_1} , increase the payment of agent a_j by the same v_{a_1} , and move item t_i from agent a_i to agent a_{i+1} for all $i = 1, \dots, j-1$. In this case, the utility of agents a_1, a_2, \dots, a_{j-1} is unchanged, the utility of agent a_j increases by $v_{a_j} - v_{a_1} > 0$, and the utility of the auctioneer is unchanged. The sum of payments by the agents is likewise unchanged. This contradicts the assumption that (M, P) is Pareto optimal.

We now seek to prove that $(R_1 \cap R_2) \Rightarrow Q$. We note above that if not all items are allocated ($\neg R_1$) then the allocation is not Pareto-optimal ($\neg Q$), thus $Q \Rightarrow R_1$ and (trivially) $Q \Rightarrow Q \cap R_1$ (Pareto optimality implies all items allocated). Thus, $(R_1 \cap R_2) \Rightarrow Q \Rightarrow Q \cap R_1$. If R_1 is false this predicate becomes $\text{False} \Rightarrow \text{False}$, thus we remain with the case where all items are allocated.

Assume $\neg Q$, i.e., assume that (M, P) is not Pareto-optimal — then there must be some other allocation (M', P') that is no worse for all players (including the auctioneer) and strictly better for at least one player. We can assume that (M', P') assigns all items as well, as otherwise we can take an even better allocation that would assign all items.

By Lemma B.1 (see below) we know that M and M' are related by a set of simple paths and cycles. On a path, the first agent gives up one item, whereas the last agent receives one item

more, after items are exchanged along the path. Cycles represents giving up one item in return for another by passing items around along it. Cycles don't change the number of items assigned to the bidders along the cycles so we will ignore them. x_1, \dots, x_z and y_1, \dots, y_z denote the start and end agents along these z alternating paths. Note that the same agent may appear multiple times amongst x_i 's or multiple times amongst y_i 's, but cannot appear both as an x_i and as a y_i (we can concatenate two such paths into one). Such an alternating path represents a shuffle of items between agents where agent x_j loses an item whereas agent y_j gains an item when moving from M to M' . In general, these two items may be entirely different.

Since there are no trading paths with respect to (M, P) , it must be the case that for every one of these z alternating paths either

- α . $v_{y_j} \leq v_{x_j}$ holds. Define $I = \{j | v_{y_j} \leq v_{x_j}\}$.
- β . $b_{y_j}^* < v_{x_j}$ holds (where $b_{y_j}^*$ is the budget left over for agent y_j at the end of the mechanism). Define $J = \{j | b_{y_j}^* < v_{x_j}\}$.

Now, no bidder is worse off in (M', P') (in comparison to (M, P)), and the auctioneer is no worse off, and, by assumption, either/or

- A. Some bidder is strictly better off. Or,
- B. The auctioneer is strictly better off.

First, we rule out case B above: Consider the process of changing (M, P) into (M', P') as a two stage process: at first, the agents x_1, \dots, x_z give up items. During this first stage, the payments made by agents x_1, \dots, x_m must decrease (in sum) by at least $Z^- = \sum_{i=1}^z v_{x_i}$. The 2nd stage is that agents y_1, \dots, y_z receive their extra items. In the 2nd stage, the maximum extra payment that can be received from agents y_1, \dots, y_z is no more than

$$Z^+ = \sum_{j \in I} v_{y_j} + \sum_{j \in J} b_{y_j}^* \leq \sum_{j \in I} v_{x_j} + \sum_{j \in J} v_{x_j} = Z^-, \quad (6)$$

by definition of sets I and J above. Thus, the total increase in revenue to the auctioneer is $Z^+ - Z^- \leq 0$. This rules out Case B above (auctioneer strictly better off). Moreover, as the auctioneer cannot be worse off, $Z^+ = Z^-$ and from Equation (6) we conclude that

$$\sum_{j \in I} v_{y_j} + \sum_{j \in J} b_{y_j}^* = \sum_{j \in I} v_{x_j} + \sum_{j \in J} v_{x_j}. \quad (7)$$

From α above, we have that $v_{y_j} \leq v_{x_j}$ for $j \in I$, from β we have that $b_{y_j}^* < v_{x_j}$ for $j \in J$. Thus, if $J \neq \emptyset$ then the lefthand side of Equation (7) is strictly less than the righthand side, a contradiction.

Therefore, case A must hold and it must be that $J = \emptyset$, we will conclude the proof of the theorem by showing that these two are inconsistent. So, we have that

$$\begin{aligned} M'_a v_a - P'_a &= M_a v_a - P_a && \text{for agents } a \text{ whose utility is unchanged} \\ M'_{\hat{a}} v_{\hat{a}} - P'_{\hat{a}} &> M_{\hat{a}} v_{\hat{a}} - P_{\hat{a}} && \text{for some agent } \hat{a} \\ \sum_a P'_a &= \sum_a P_a. \end{aligned}$$

We can now derive that

$$\begin{aligned}
\sum_a M'_a v_a &> \sum_a M_a v_a - \left(\sum_a P'_a - \sum_a P_a \right) \\
&= \sum_a M_a v_a. \\
\Rightarrow \sum_a (M'_a - M_a) v_a &> 0.
\end{aligned} \tag{8}$$

Now, whenever $a = x_j$ we decrease $M'_a - M_a$ by one, whenever $a = y_j$ we increase $M'_a - M_a$ by one. Thus, rewriting Equation (8) we get that

$$\begin{aligned}
\sum_a (|\{j|a = y_j\}| - |\{j|a = x_j\}|) v_a &> 0 \\
\Rightarrow \sum_{j=1}^z v_{y_j} - \sum_{j=1}^z v_{x_j} &> 0 \\
\Rightarrow \sum_{j=1}^z v_{y_j} &> \sum_{j=1}^z v_{x_j}.
\end{aligned} \tag{9}$$

But, Equation (9) is inconsistent with Equation (7) as $J = \emptyset$ implies that $I = \{1, \dots, z\}$. \square

The following technical lemma was required in the proof of Theorem 4.5 above:

Lemma B.1 *Let M and M' be two B -matchings that allocate all items, then, the symmetric difference between these two matchings, $M \oplus M'$, can be decomposed into a set of simple alternating paths (with respect to M) and alternating cycles (also with respect to M) that are edge disjoint. Moreover, there are no two simple alternating paths such that one ends and the other begins at the same agent.*

Proof: Intuitively, the set $M \oplus M'$ relates M to M' and shows how to change one matching into another. To prove the lemma, direct edges in M from agents to items and edges in M' from items to agents. Denote the resulting graph as \vec{G} . Any directed graph (and \vec{G} in particular) can be decomposed into a set of simple paths and cycles, such that no two simple paths start and end in the same vertex, i.e., maximal length simple paths.

To prove that such paths cannot start or end at an item, recall that both M and M' allocate all items. Thus, every item is adjacent to one edge in M and one edge in M' , so in $M \oplus M'$ it is adjacent to either zero or to 2 edges. Should we assume that some path starts at an item, this contradicts our assumption of maximal paths in \vec{G} . A similar argument shows that no path can end at an item. Therefore, all paths start and end at an agent. The maximality of the paths in \vec{G} also shows that there are no two paths such that one ends and the other begins at the same agent.

Along any such path or cycle, there can be no two consecutive edges from M and there can be no two consecutive edges from M' . Also, for all edges in $M \oplus M'$ between an agent i and an item j , it must be that $j \in S_i$. Thus all maximal paths and all cycles covering \vec{G} are alternating paths with respect to M . We also remark that should we reverse the direction of the paths and cycles then they will be alternating paths with respect to M' .

□

C Discussion and Remarks

We hope that the following remarks may prove helpful:

1. In the definition of Pareto-optimality (Definition 4.1), one allows any alternative allocation and pricing. If (for example) we were to redefine Pareto optimality, defining “Pareto-optimality” by appending to the sentence fragment “for no other allocation (M', P') ” the suffix “*such that $P'_i \geq 0$ for all i* ”. Then, “Pareto-optimal” assignments could in fact contain trading paths. Such trades would be “illegal” because they would violate the no positive transfers condition ($P'_i \geq 0$).
2. Pareto-optimality as given in Definition 4.1 is a more desirable social goal than “Pareto-optimal” invented above. If we only insisted on a “Pareto-optimal” assignment, then we could get very bad assignments. Later, subsequent to the auction, the bidders could trade amongst themselves and improve their lot.
3. However, it may also be desirable that no agent actually get paid from the mechanism. Thus, it may be desirable that the actual allocation produced by the action have no positive transfers ($P_i \geq 0$ for all i), yet at the same time be Pareto-optimal in the strong sense of Definition 4.1: after the allocation is presented, no agents will desire to trade amongst themselves. This is the claim of Theorem 4.2.

D Proof of Theorem 5.1

For the proof of Theorem 5.1 (up to by not including Corollary 5.1) we assume the step function valuations (as done throughout this paper).

Recall the uniqueness result of [3]:

Theorem D.1 (Theorem 5.1 of [3]) *Let A be a truthful, bidder-rational, auctioneer rational, and Pareto-optimal multi unit auction (identical items) with 2 players with known (public) budgets b_1, b_2 that are generic⁶ then if $v_1 \neq v_2$ the allocation produced by A is identical to that produced by the Dynamic clinching auction of [3] (and, in particular, with our auction when applied to these inputs).*

For all the details of the proof please see [4], as the original publication [3] includes only a sketch.

D.1 Public budgets b_i , Private valuations v_i , and Private sets of interest S_i

We now show that there is no incentive compatible, Pareto-optimal, bidder rational, and auctioneer rational mechanism when the budgets are public, and the agent valuation and set of interest is private.

We say that an agent wins an item if the item is assigned to the agent.

Consider two agents, 1 and 2, and two items t_1, t_2 . Let $S_1 = \{t_1\}$ and $S_2 = \{t_1, t_2\}$. We now prove the following:

⁶Not all pairs of values are generic, but for our purposes assume that this holds for every such pair.

Lemma D.2 *Consider any incentive compatible, Pareto-optimal, bidder rational and auctioneer rational combinatorial auction that produces an allocation (M, P) : if agent 2 wins both items then the payment P_1 by agent 1 is zero.*

Proof: First, consider the case when $v_1 = 0$. Then any incentive compatibility and Pareto-optimality auction has to assign both items to agent 2. If any of the items were to be left unassigned, or would be assigned to agent 1, we could assign it to agent 2, without changing any payment. This does not change the utility of agent 1, nor the utility of the auctioneer, but would strictly increase the utility of agent 2.

Because of incentive compatibility, agent 2 pays $P_2 = 0$. Otherwise, agent 2 could reduce his reported valuation and attain the item at a lower price. It follows from bidder rationality that $P_1 \leq 0$ (we have not ruled out positive transfers yet). However, it follows from auctioneer rationality that agent one must pay zero, as $-P_1 \leq P_2 = 0$.

Now, consider the case when both agents have nonzero valuations. Then for every instance in which agent 1 gets no items it must be that $P_1 = 0$. By IC his payment cannot depend on his valuation, and when agent 1 reported a valuation of zero then P_1 was zero. \square

Lemma D.3 *Consider any incentive compatible, Pareto-optimal, bidder rational and auctioneer rational combinatorial auction that produces an allocation (M, P) : if agent 2 does not win item t_1 then $P_2 = 0$.*

Proof: First consider the case when $v_2 = 0$, and $v_1 > 0$. As in previous proof, any incentive compatibility and Pareto-optimality auction has to assign item t_1 to agent 1. It follows from incentive compatibility that agent 1 pays $P_1 = 0$, whereas it follows from bidder rationality and auctioneer rationality that $P_2 = 0$.

Now, consider the case when both agents have nonzero valuations. On every input when agent 2 is not assigned item t_1 , it must be that $P_2 = 0$, this follows since by incentive compatibility P_2 cannot depend on v_2 . \square

The lemmata above allow us to argue about payment, but don't tell us which matching is chosen. This is done in the following lemma.

Lemma D.4 *If $b_1 < b_2$, $b_1 < v_2$, and $v_1 \neq v_2$, then any incentive compatible, Pareto-optimal, bidder rational and auctioneer rational combinatorial auction has to assign both items to agent 2.*

Proof: We want to show that independently of what agent 1 says (but $v_1 \neq v_2$), agent 2 will get both items.

We first concentrate on the case when $v_1 \leq b_2$. Observe that the only PO allocation assigns both items to agent 2. By Lemma D.3, if item t_1 was allocated to agent 1 then $P_2 = 0$. In this case player 2 can buy the item from 1 and they are both better off.

Now, consider the case when $b_1 < v_1 < b_2$. By the above argument player 1 cannot be allocated item t_1 . Suppose that for some value $v'_1 > b_1$ the allocation assigns item t_1 to agent 1. even though $v_2 > b_1$ and $b_2 > b_1$. As agent 1 is never charged more than her budget, $P_1 \leq b_1$. Then the utility for agent 1 is $v_1 - b_1 > 0$: agent 1 has incentive to lie about v_1 , contradicting IC.

Hence, there is no value $v'_1 > b_1$ such that if agent 1 claims a valuation of v'_1 then the mechanism assigns t_1 to agent 1. This in turn implies that even if the truth is that $v_1 > b_2$, player 2 must still be assigned both items t_1 and t_2 . \square

We are now ready to prove the main result of this section.

Theorem D.5 *There is no incentive compatible, Pareto-optimal, bidder rational and auctioneer rational combinatorial auction with public budgets, b_a , private valuations, v_a , and private sets of interest, S_a .*

Proof: Consider the case of two agents, 1 and 2, and two items t_1, t_2 . Let $S_1 = \{t_1, t_2\}$ and $S_2 = \{t_1, t_2\}$. Additionally, Fix $v_1 = 10$, $v_2 = 11$, $b_1 = 4$ and $b_2 = 5$. In this case, by Theorem D.1, the allocation must coincide with the result of the dynamic clinching auction of [3].

I.e., both agents get one of the two items, $p_1 = 3$, and $p_2 = 2$. Without loss of generality assume that item t_1 is assigned to agent 1 with probability at least $\frac{1}{2}$ (if the mechanism is randomized).

Now, assume that the true set of interest for agent 1 was in fact $S_1 = \{t_1\}$. We argue that agent 1 now has incentive to lie about S_1 :

- if agent 1 reports her true set of interest – then by Lemma D.4 both items end up assigned to agent 2, and by Lemma D.2 $P_1 = 0$, so her utility is zero as well;
- if agent 1 lies and reports $\{t_1, t_2\}$ as her set of interest – then with probability $\leq \frac{1}{2}$ her utility is equal to $0 - 3$, and with probability at least $\frac{1}{2}$ her utility is equal to $10 - 3 = 7$, so on average his utility is at least $-3 \cdot \frac{1}{2} + 7 \cdot \frac{1}{2} = 2$.

This concludes the proof as agent 1 has incentive to lie in any incentive compatible, Pareto-optimal, bidder rational and auctioneer rational combinatorial auction. \square